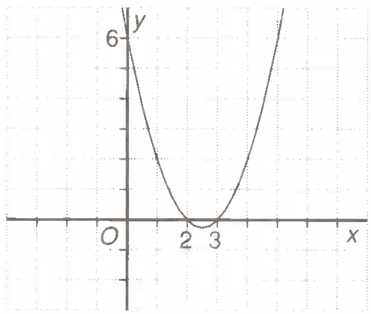
In Search of Graphical Representation of Non-real Solutions of Equations

Solutions to quadratic equations in $\mathbb{R}$ can be seen graphically as shown in the examples A and B that follow.

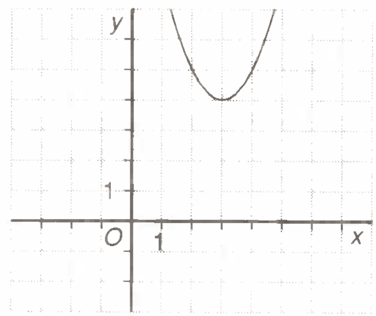
A. $x^2 - 5x + 6 = 0$
$\Rightarrow (x - 3)(x - 2) = 0$
$\Rightarrow x = 3$ or $x = 2$.

Graphically, these solutions can be viewed as the points where the parabola $y = x^2 - 5x + 6$ ① intersects with the line $y = 0$, that is, the x-axis.



B. $x^2 - 6x + 13 = 0$ has no real solutions.
The quadratic formula yields $x = 3 + 2i$ or $x = 3 - 2i$.

The parabola $y = x^2 - 6x + 13$ ② does *not* intersect the x-axis.
Is there any geometric significance in this context for $3 + 2i$ and $3 - 2i$?



Pursuing the question asked in B, proceed as follows. Allow the x-values in the parabola ② to extend into $\mathbb{C}$, that is, let x take the form $a + bi$, with $a, b \in \mathbb{R}$.

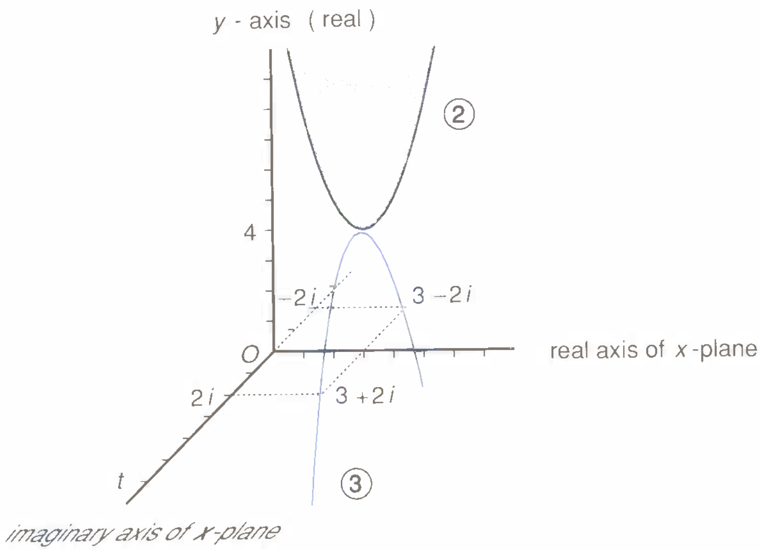
You now have a complex plane, the x-plane, taking the place of the old x-axis. (Note that the old x-axis is contained in this complex plane.)

Unfortunately, y will also now take on non-real values, and a four-dimensional situation is set up.

However, it is still possible to view a part of this, as follows.

You already know that the solutions to $y = 0$ are $x = 3 \pm 2i$. Hence, the real part of each solution is 3. Allow x to take the form $3 + ti$, with $t \in \mathbb{R}$.

Then $y = (3 + ti)^2 - 6(3 + ti) + 13$
$\quad = 9 + 6ti - t^2 - 18 - 6ti + 13$
or $\quad y = 4 - t^2$ ③



Now t is the variable along the imaginary axis of the x-plane.

The equation ③ thus represents a parabola whose plane is perpendicular to the plane of the original parabola ②.

Also, this parabola punctures the x-plane at the points $3 + 2i$ and $3 - 2i$.

Thus, you can see that the non-real intersections of a parabola with the x-axis are "somewhere in front of, or behind, the paper"!

A Canadian mathematician, Richard Dewsbury, is presently researching the geometrical aspect of extensions to $\mathbb{C}$ of equations in $\mathbb{R}$.